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Fractional Lévy motion through path integrals

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Abstract

Fractional Lévy motion (fLm) is the natural generalization of fractional Brownian motion in the context of self-similar stochastic processes and stable probability distributions. In this paper we give an explicit derivation of the propagator of fLm by using path integral methods. The propagators of Brownian motion and fractional Brownian motion are recovered as particular cases. The fractional diffusion equation corresponding to fLm is also obtained.

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1. Introduction

It is widely known that if $\xi_2(t)$ is a Gaussian, uncorrelated noise (i.e. white noise) the Langevin (stochastic) equation

$$x(t) = x_0 + \int_0^t \xi_2(t')\,\mathrm{d}t' \tag{1}$$

describes ordinary Brownian motion. One of the properties of Brownian motion is that the average squared displacement grows linearly with time, $\langle (x(t) - x_0)^2\rangle \propto t$. However, many transport processes in physical, biological and social systems exhibit anomalous diffusion [1, 2]. That is, $\langle (x(t) - x_0)^2\rangle \propto t^{2H}$, with $H \neq 1/2$, where H is called the Hurst exponent [3]. In the past, several authors have attempted to generalize equation (1) in order to accommodate these anomalous processes. The anomalous behaviour may be associated with the existence of spatiotemporal correlations that produce correlated increments $\mathrm{d}x(t) := x(t + \mathrm{d}t) - x(t)$. For this reason, the first proposed generalization had the form

$$x(t) = x_0 + \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - t')^{H-1/2} \xi_2(t')\,\mathrm{d}t', \tag{2}$$

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known as fractional Brownian motion (fBm) [4], which has been extensively studied and applied [5–8]. In equation (2) the noise $\xi_2(t)$ remains Gaussian and uncorrelated but, thanks to the convolution with the power-law kernel, the increments $\mathrm{d}x(t)$ become correlated in a way that yields the desired scaling of the mean squared displacement as well as other self-similar properties. In particular, the average motion remains invariant under the transformation $(x, t) \mapsto (\lambda^H x, \lambda t)$, generalizing the self-similarity of the original Brownian motion, which is obviously recovered when $H = 1/2$ is set in equation (2). H is the self-similarity exponent of the process. When rewritten in terms of the Riemann–Liouville fractional integral operators (see the appendix) fBm reads

$$x(t) = x_0 + {}_0D_t^{-(H+1/2)}\xi_2. \tag{3}$$

The corresponding propagator and diffusion equation of fBm have been derived in a number of ways in the literature (see, for example, [9, 10]).

We will devote this paper to the Lévy generalization of fractional Brownian motion, known as fractional Lévy motion (fLm) [11–14]. The Langevin equation defining the process is

$$x(t) = x_0 + {}_0D_t^{-H+1/\alpha-1}\xi_\alpha, \tag{4}$$

where $\xi_\alpha(t)$ is time uncorrelated and distributed, for each t, according to a symmetric Lévy distribution [15]. We recall here that symmetric Lévy distributions are the symmetric solutions of the generalized central limit theorem and are parametrized by the stability index, $\alpha \in (0, 2]$, and the scale factor, $\sigma > 0$. The characteristic function (i.e. the Fourier transform) of a symmetric Lévy distribution $L_{\alpha,\sigma}(u)$ is

$$\mathcal{F}[L_{\alpha,\sigma}](k) = \exp(-\sigma^\alpha |k|^\alpha). \tag{5}$$

In particular, for $\alpha \in (0, 2)$, the Lévy distributions have algebraic tails,

$$L_{\alpha,\sigma}(u) \sim \frac{C_\alpha}{|u|^{\alpha+1}}, \qquad |u| \to \infty. \tag{6}$$

A Lévy distribution with $\alpha = 2$ is a Gaussian,

$$L_{2,\sigma}(u) = \frac{1}{2\sigma\sqrt{\pi}} \exp\left(-\frac{u^2}{4\sigma^2}\right), \tag{7}$$

and σ is related to the second moment, $\langle u^2\rangle = 2\sigma^2$.

The need for fLm originates in the observation of power-law (Lévy) statistics for the displacements $\mathrm{d}x(t)$ in many physical systems of interest [1, 2], which is in contrast to the Gaussian character of fBm. In equation (4), H is still the self-similarity exponent of the process. It can be shown [15] that finiteness requirements for certain moments of x restrict the admissible values of H to

$$H \in \begin{cases} \left(0, \frac{1}{\alpha}\right] & 0 < \alpha \leqslant 1 \\ (0, 1] & 1 < \alpha \leqslant 2. \end{cases} \tag{8}$$

Under these circumstances the sth moment of x, with $0 < s < \alpha$, behaves as $\langle |x|^s\rangle \propto t^{sH}$.

Note that fBm is recovered from equation (4) if $\alpha = 2$. Also, the increments $\mathrm{d}x(t)$ of the process are uncorrelated for $H = 1/\alpha$. In this case, ordinary Lévy motion is obtained, of which Brownian motion is a particular case ($H = 1/\alpha = 1/2$). In the present work we will compute in detail the propagator of fLm through path-integral techniques [16], now familiar in both quantum field theory and statistical physics. Although the form of this propagator has previously been derived in a more abstract way by using self-similarity and stability arguments [11], the path-integral calculation offers a new insight which might help extend the range of applications of fLm and even tackle more complicated situations. Two of us recently computed the propagator of fBm by path integral methods [17] taking advantage of the fact that the measure can be written as the exponential of an action quadratic in the fields. As we will see the fLm path-integral measure is not Gaussian in the fields (except for $\alpha = 2$) and the computation becomes quite more involved.

The rest of the paper is organized as follows. In section 2 we construct the appropriate probability measure on the space of realizations of the noise for fLm. The propagator is defined as a particular expectation value. Section 3 gives a detailed calculation of the propagator of fLm in the path integral formalism. In section 4 the fractional diffusion equation satisfied by the propagator of fLm is worked out. Section 5 contains the conclusions. The appendix collects some basic definitions on fractional integrals and derivatives.

2. Construction of the path integral measure

Assume that the motion of a particle is defined by a stochastic differential equation. The propagator $G(x_T, T|x_0, 0)$ is, by definition, the probability of finding the particle at $x = x_T$ at time $t = T$ if initially, $t = 0$, it was located at $x = x_0$. As mentioned above, the main objective of this paper is to compute the propagator associated with equation (4) (we drop the subscript α of ξ from now on) by means of path integrals. Consider trajectories $x(t) : [0, T] \to \mathbb{R}$ with boundary conditions $x(0) = x_0$ and $x(T) = x_T$. From equation (4) we immediately deduce that the boundary conditions of $x(t)$ are translated into the following constraint on $\xi(t)$:

$$ {}_0D_T^{-H+1/\alpha-1}\xi = x_T - x_0. \tag{9}$$

The essential object in the path integral formalism is the probability measure $\mathcal{P}(\xi(t))\mathcal{D}\xi(t)$ on the space of realizations of the noise, i.e. on the space of maps $\xi(t) : [0, T] \to \mathbb{R}$. Once it is constructed, the propagator is defined as the following expectation value:

$$ G(x_T, T|x_0, 0) = \int \delta\big({}_0D_T^{-H+1/\alpha-1}\xi - (x_T - x_0)\big)\mathcal{P}(\xi(t))\mathcal{D}\xi(t). \tag{10}$$

In order to construct the measure associated with the Langevin equation (4) we will discretize the time at $N+1$ points $t_n := n\epsilon, n = 0, 1, \ldots, N$, with $\epsilon := T/N$. The continuum limit, $N \to \infty$, will be taken eventually. Each path is discretized according to $x_n := x(t_n)$. The appropriate discretization of the noise is made by taking $\xi(t_n) = \epsilon^{-1+1/\alpha}\xi_n$, where each ξ_n is an independent random variable distributed according to a symmetric Lévy distribution of index α. The factor $\epsilon^{-1+1/\alpha}$ ensures the correct time dependence of the finite moments of x, $\langle |x|^s\rangle \propto t^{sH}, 0 < s < \alpha$. Therefore, the probability measure is naturally defined as

$$ \mathcal{P}(\xi(t))\mathcal{D}\xi(t) = \prod_{n=1}^{N} L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n. \tag{11}$$

Using the definition of the fractional integral, equation (A.1), the constraint (9) can be written as

$$ \frac{1}{\Gamma(H - 1/\alpha + 1)}\int_0^T (T-\tau)^{H-1/\alpha}\xi(\tau)\,\mathrm{d}\tau = x_T - x_0. \tag{12}$$

Discretizing as prescribed above:

$$ \begin{aligned} \int_0^T (T-\tau)^{H-1/\alpha}\xi(\tau)\,\mathrm{d}\tau &= \sum_{n=1}^{N}\xi(n\epsilon)\int_{(n-1)\epsilon}^{n\epsilon}(T-\tau)^{H-1/\alpha}\mathrm{d}\tau \\ &= \frac{\epsilon^H}{H - 1/\alpha + 1}\sum_{n=1}^{N}\xi_n[(N-n+1)^{H-1/\alpha+1} - (N-n)^{H-1/\alpha+1}], \end{aligned} \tag{13}$$

and we can easily solve for ξ_N in terms of ξ_n, $n = 1, \ldots, N-1$:

$$\xi_N = A - \sum_{n=1}^{N-1} B_n \xi_n, \tag{14}$$

with

$$A := \frac{\Gamma(H - 1/\alpha + 2)}{\epsilon^H}(x_T - x_0),$$
$$B_n := (N-n+1)^{H-1/\alpha+1} - (N-n)^{H-1/\alpha+1}, \qquad n = 1, \ldots, N-1. \tag{15}$$

Now, we are ready to write an explicit expression for the expectation value defining the propagator (10). Namely,

$$G(x_T, T|x_0, 0) = \lim_{N\to\infty} f(T, N) \int \delta\left(\xi_N - A + \sum_{n=1}^{N-1} B_n \xi_n\right) \prod_{n=1}^{N} L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n, \tag{16}$$

where $f(T, N)$ is a normalization factor which will be determined at the end of the calculation. Equivalently, integrating over ξ_N in equation (16):

$$G(x_T, T|x_0, 0) = \lim_{N\to\infty} f(T, N) \int L_{\alpha,\sigma}\left(A - \sum_{n=1}^{N-1} B_n \xi_n\right) \prod_{n=1}^{N-1} L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n. \tag{17}$$

It is instructive to show that the path integral of ordinary Brownian motion, usually introduced in a different fashion, coincides with equation (16) when $H = 1/\alpha = 1/2$. The Langevin equation for Brownian motion is (recall equation (1))

$$\dot{x}(t) = \xi_2(t). \tag{18}$$

The propagator is customarily introduced as

$$G(x_T, T|x_0, 0) = \int \delta(x(0) - x_0)\,\delta(x(T) - x_T) \exp\left(-\frac{1}{4\sigma^2}\int_0^T \dot{x}(t)^2 \mathrm{d}t\right) \mathcal{D}x(t), \tag{19}$$

where the paths are weighted by the classical action of the free particle. Now, one can choose the velocity, $v(t) = \dot{x}(t)$, as the integration variable. The transformation is linear and the Jacobian does not depend on the fields. The boundary conditions are simply translated into

$$\int_0^T v(t)\,\mathrm{d}t = x_T - x_0. \tag{20}$$

Therefore, we can write

$$G(x_T, T|x_0, 0) = \int \delta\left(\int_0^T v(t)\,\mathrm{d}t - (x_T - x_0)\right) \exp\left(-\frac{1}{4\sigma^2}\int_0^T v(t)^2 \mathrm{d}t\right) \mathcal{D}v(t). \tag{21}$$

If we discretize the paths as above we get

$$G(x_T, T|x_0, 0) = \lim_{N\to\infty} f(T, N) \int \delta\left(\epsilon \sum_{n=1}^{N} v_n - (x_T - x_0)\right) \prod_{n=1}^{N} \exp\left(-\frac{\epsilon}{4\sigma^2} v_n^2\right) \mathrm{d}v_n, \tag{22}$$

where $f(T, N)$ is a normalization factor. Finally, with a last change of variables, $\xi_n := \epsilon^{1/2} v_n$ (redefine $f(T, N)$ as needed):

$$G(x_T, T|x_0, 0) = \lim_{N\to\infty} f(T, N) \int \delta\left(\sum_{n=1}^{N} \xi_n - \frac{x_T - x_0}{\epsilon^{1/2}}\right) \prod_{n=1}^{N} \exp\left(-\frac{1}{4\sigma^2}\xi_n^2\right) \mathrm{d}\xi_n, \tag{23}$$

which is exactly equation (16) for $H = 1/\alpha = 1/2$ (recall equation (7)).

3. Path integral computation of the propagator of fractional Lévy motion

The computation of (17) is performed by repeated use of the identity

$$\int_{-\infty}^{\infty} L_{\alpha,\sigma}(x)L_{\alpha,\sigma}(y-\lambda x)\,\mathrm{d}x = (1+\lambda^{\alpha})^{-1/\alpha}L_{\alpha,\sigma}\left(\frac{y}{(1+\lambda^{\alpha})^{1/\alpha}}\right), \qquad \forall\, y\in\mathbb{R}, \tag{24}$$

where $L_{\alpha,\sigma}(x)$ is a symmetric Lévy distribution with index α and scale factor σ, and λ is a positive real number. The proof is straightforward. Define $\bar{L}_{\alpha,\sigma}(x) := L_{\alpha,\sigma}(\lambda x)$. Then,

$$\int_{-\infty}^{\infty} L_{\alpha,\sigma}(x)L_{\alpha,\sigma}(y-\lambda x)\,\mathrm{d}x = \mathcal{F}^{-1}[\mathcal{F}[L_{\alpha,\sigma}]\mathcal{F}[\bar{L}_{\alpha,\sigma}]](\lambda^{-1}y). \tag{25}$$

Using that $\mathcal{F}[\bar{L}_{\alpha,\sigma}](k) = |\lambda|^{-1}\mathcal{F}[L_{\alpha,\sigma}](k/\lambda)$ and $\mathcal{F}[L_{\alpha,\sigma}](k) = \exp(-\sigma^{\alpha}|k|^{\alpha})$ we get:

$$\mathcal{F}[L_{\alpha,\sigma}](k)\mathcal{F}[\bar{L}_{\alpha,\sigma}](k) = |\lambda|^{-1}\exp(-\sigma^{\alpha}|(1+|\lambda|^{-\alpha})^{1/\alpha}k|^{\alpha}). \tag{26}$$

And equation (24) follows easily.

Let us go back to equation (17). Using equation (24) we integrate out ξ_1:

$$G(x_T,T|x_0,0) = \lim_{N\to\infty} f(T,N)\int L_{\alpha,\sigma}\left(\frac{A-\sum_{n=2}^{N-1}B_n\xi_n}{\left(1+B_1^{\alpha}\right)^{1/\alpha}}\right)\prod_{n=2}^{N-1}L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n. \tag{27}$$

Integration of ξ_2 yields

$$\begin{aligned} G(x_T,T|x_0,0) &= \lim_{N\to\infty} f(T,N)\int L_{\alpha,\sigma}\left(\frac{A-\sum_{n=3}^{N-1}B_n\xi_n}{\left(1+B_1^{\alpha}\right)^{1/\alpha}\left(1+\frac{B_2^{\alpha}}{1+B_1^{\alpha}}\right)^{1/\alpha}}\right)\prod_{n=3}^{N-1}L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n \\ &= \lim_{N\to\infty} f(T,N)\int L_{\alpha,\sigma}\left(\frac{A-\sum_{n=3}^{N-1}B_n\xi_n}{\left(1+B_1^{\alpha}+B_2^{\alpha}\right)^{1/\alpha}}\right)\prod_{n=3}^{N-1}L_{\alpha,\sigma}(\xi_n)\,\mathrm{d}\xi_n. \end{aligned} \tag{28}$$

And after $N-1$ integrations:

$$G(x_T,T|x_0,0) = \lim_{N\to\infty} f(T,N)L_{\alpha,\sigma}\left(\frac{A}{\left(1+\sum_{n=1}^{N-1}B_n^{\alpha}\right)^{1/\alpha}}\right). \tag{29}$$

It remains to compute

$$\begin{aligned} \lim_{N\to\infty}\frac{A}{\left(1+\sum_{n=1}^{N-1}B_n^{\alpha}\right)^{1/\alpha}} &= \lim_{N\to\infty}\Gamma(H-1/\alpha+2)\frac{x_T-x_0}{T^H} \\ &\quad\times N^H\left[1+\sum_{n=1}^{N-1}((N-n+1)^{H-1/\alpha+1}-(N-n)^{H-1/\alpha+1})^{\alpha}\right]^{-1/\alpha}. \end{aligned} \tag{30}$$

In the following, $g(N)\sim h(N)$ will mean that $g(N)/h(N)\to 1$ when $N\to\infty$. First observe that

$$\sum_{n=1}^{N-1}((N-n+1)^{H-1/\alpha+1}-(N-n)^{H-1/\alpha+1})^{\alpha} \sim (H-1/\alpha+1)^{\alpha}N^{\alpha H-1}\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)^{\alpha H-1}, \tag{31}$$

where we have used

$$\left(1-\frac{n}{N}+\frac{1}{N}\right)^{H-1/\alpha+1} \sim \left(1-\frac{n}{N}\right)^{H-1/\alpha+1}+\frac{H-1/\alpha+1}{N}\left(1-\frac{n}{N}\right)^{H-1/\alpha}. \tag{32}$$

Now, we note that

$$\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)^{\alpha H-1}\frac{1}{N}\sim\int_{1/N}^{1-1/N}(1-u)^{\alpha H-1}\mathrm{d}u=\frac{1}{\alpha H}\left[\left(1-\frac{1}{N}\right)^{\alpha H}-\frac{1}{N^{\alpha H}}\right],\tag{33}$$

and we have almost reached our goal. Combining equations (31) and (33) we get

$$\sum_{n=1}^{N-1}((N-n+1)^{H-1/\alpha+1}-(N-n)^{H-1/\alpha+1})^{\alpha}\sim\frac{(H-1/\alpha+1)^{\alpha}}{\alpha H}N^{\alpha H}.\tag{34}$$

Inserting this in equation (30):

$$\lim_{N\to\infty}\frac{A}{\left(1+\sum_{n=1}^{N-1}B_n^{\alpha}\right)^{1/\alpha}}=(\alpha H)^{1/\alpha}\Gamma(H-1/\alpha+1)\frac{x_T-x_0}{T^H}.\tag{35}$$

Hence,

$$G(x_T,T|x_0,0)=f(T)L_{\alpha,\sigma}\left((\alpha H)^{1/\alpha}\Gamma(H-1/\alpha+1)\frac{x_T-x_0}{T^H}\right),\tag{36}$$

and $f(T)$ can be determined by normalization, $\int_{-\infty}^{\infty}G(x_T,T|x_0,0)\,\mathrm{d}x_T=1$:

$$f(T)=\frac{(\alpha H)^{1/\alpha}\Gamma(H-1/\alpha+1)}{T^H},\tag{37}$$

so that the final expression of the propagator is

$$G(x_T,T|x_0,0)=\frac{(\alpha H)^{1/\alpha}\Gamma(H-1/\alpha+1)}{T^H}L_{\alpha,\sigma}\left((\alpha H)^{1/\alpha}\Gamma(H-1/\alpha+1)\frac{x_T-x_0}{T^H}\right).\tag{38}$$

Summarizing, we have deduced that the propagator of fLm is a Lévy distribution depending on the combination x/t^H, so that the average motion is self-similar with exponent H.

4. Fractional diffusion equation

For the sake of completeness we derive in this section the fractional diffusion equation which governs the time evolution of the propagator of fLm. Denote by $\hat{G}(k,t)$ the Fourier transform of $G(x,t|x_0,0)$ with respect to x. Using equation (38), the form of the characteristic function of a Lévy distribution, and the properties of the Fourier transform under rescaling:

$$\hat{G}(k,t)=\exp\left(-\frac{\sigma^{\alpha}t^{\alpha H}}{(\alpha H)\Gamma^{\alpha}(H-1/\alpha+1)}|k|^{\alpha}\right).\tag{39}$$

Differentiating with respect to t:

$$\frac{\partial}{\partial t}\hat{G}(k,t)=-\frac{\sigma^{\alpha}t^{\alpha H-1}}{\Gamma^{\alpha}(H-1/\alpha+1)}|k|^{\alpha}\hat{G}(k,t).\tag{40}$$

Fourier inverting, recalling the definition (A.3) and the identity

$$\mathcal{F}\left[\frac{\partial^{\alpha}f}{\partial|x|^{\alpha}}\right](k)=-|k|^{\alpha}f(k),\tag{41}$$

we find:

$$\frac{\partial}{\partial t}G(x,t)=\frac{\sigma^{\alpha}t^{\alpha H-1}}{\Gamma^{\alpha}(H-1/\alpha+1)}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}G(x,t).\tag{42}$$

Therefore, the propagator of fLm satisfies a space-fractional diffusion equation with time-dependent diffusivity. Equation (42) was recently derived by different methods in [18].

The equation for the propagator of fBm (appeared in [10]) is obtained from equation (42) in the particular case $\alpha = 2$:

$$\frac{\partial}{\partial t} G(x,t) = \frac{\sigma^2 t^{2H-1}}{\Gamma^2(H+1/2)} \frac{\partial^2}{\partial x^2} G(x,t), \tag{43}$$

which is a diffusion equation with time-dependent diffusivity.

Finally, if $H = 1/\alpha = 1/2$ we retrieve the standard diffusion equation associated with ordinary Brownian motion,

$$\frac{\partial}{\partial t} G(x,t) = \sigma^2 \frac{\partial^2}{\partial x^2} G(x,t). \tag{44}$$

5. Conclusions

The Langevin equation defining fLm consists of two main ingredients: a time-uncorrelated stochastic noise distributed according to a Lévy distribution and a fractional integral operator which generates the time correlations. In this paper, we have derived the propagator of fLm (which was deduced in [11] by using self-similarity arguments) through path integral techniques. That is, we have explicitly constructed a probability measure on the set of realizations of the noise and precisely defined the propagator as an average over this measure space. The computation of the propagator has been performed by discretizing the paths and carefully taking the continuum limit at the end. The fractional diffusion equation associated with fLm has also been derived. We hope that the heuristic power of the path integral formalism will provide new insight into the calculation and help address more complicated cases.

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Appendix. Riemann–Liouville fractional integral and differential operators

The books [19, 20] are excellent introductory texts to fractional calculus containing, in particular, the following definitions.

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently well-behaved function. The *Riemann–Liouville fractional integral operators* of order α are defined as

$$\begin{aligned} {}_aD_x^{-\alpha} f &:= \frac{1}{\Gamma(\alpha)} \int_a^x (x-x')^{\alpha-1} f(x')\, \mathrm{d}x', \\ {}^bD_x^{-\alpha} f &:= \frac{1}{\Gamma(\alpha)} \int_x^b (x'-x)^{\alpha-1} f(x')\, \mathrm{d}x'. \end{aligned} \tag{A.1}$$

As for the *Riemann–Liouville fractional differential operators* of order α, the definition is

$$\begin{aligned} {}_aD_x^{\alpha} f &:= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_a^x \frac{f(x')}{(x-x')^{\alpha-m+1}}\, \mathrm{d}x', \\ {}^bD_x^{\alpha} f &:= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_x^b \frac{f(x')}{(x'-x)^{\alpha-m+1}}\, \mathrm{d}x', \end{aligned} \tag{A.2}$$

where m is the integer number verifying $m - 1 \leqslant \alpha < m$.

Finally, the Riesz fractional differential operator is defined as the symmetric combination

$$\frac{\partial^\alpha}{\partial|x|^\alpha} := \frac{-1}{2\cos(\pi\alpha/2)}\left({}_{-\infty}D_x^\alpha + {}^{\infty}D_x^\alpha\right). \tag{A.3}$$

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